Discrete Mathematics 180 (1998) 315–343

Cycle type and descent set in wreath products[1]

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Received 18 July 1995; received in revised form 29 January 1996; accepted 15 July 1996

Abstract

We express the number of elements of the hyperoctahedral group B_n, which have descent set K and such that their inverses have descent set J, as a scalar product of two representations of B_n. We also give the number of elements of B_n, which have a prescribed descent set and which are in a given conjugacy class of B_n by another scalar product of representations of B_n. For this, we first establish corresponding results for certain wreath products. We finally give, by generating series of symmetric functions, some analogs of the classical formulas which express the exponential generating series of alternating elements in the B_n's.

1. Introduction

Enumerating permutations according to certain statistics, such as descent set, major index or cycle type, is an old problem (see [1,18]). In [23], Solomon defines, for each subset K of $\{1,\dots,n-1\}$, a representation ψ_K of S_n such that the dimension of ψ_K is the number of permutations in S_n with descent set K. The characteristic symmetric function of this representation appears already in MacMahon's work [18]. See also [14].

In [21,3] appear representations X_λ of S_n, indexed by the partitions λ of n, such that the number of permutations of cycle type λ is the dimension of X_λ.

Moreover, Foulkes [8] and Gessel [9] have proved that the number of permutations σ in S_n, with descent set $K \subseteq \{1,\dots,n-1\}$ and such that the descent set of σ^{-1} is $J \subseteq \{1,\dots,n-1\}$, is the scalar product $\langle \psi_K, \psi_J \rangle$. Gessel and Reutenauer have shown in [10] a related result giving the number of permutations with descent set $\{1,\dots,n-1\}\backslash K$ and with cycle type λ as the scalar product $\langle \psi_K, X_\lambda \rangle$. For very precise enumerating results, one can also see [5].

The literature also furnishes extensions of the enumeration of certain permutations to Coxeter groups (see [2,23,25]) or to wreath products (see [6,20,26]). As for it, the

[1] Expanded version of the talk: Cycle type and descent set in hyperoctahedral groups.
[2] Supported by ISM (Montréal).

PII S0012-365X(97)00123-4

study of representations of wreath products goes back to the 1930s (see [24]) and has been recently developed in [13,28].

In this article, we extend previous results to the context of wreath products and to the particular case of hyperoctahedral groups.

Section 2 is devoted to the study of a generalisation of the characteristic function of Frobenius. This function is slightly different than the one of Macdonald. See [17] and the second edition of [16]. We also give an extension of Schur–Weyl duality (Theorem 6). A proof of this extension can be found in [19]. These two notions are our main tools.

In Section 3, we define the signed descent set of an element of a certain wreath product G_n. With the help of quasi-symmetric functions (which extend those of [9]), we can express the number of elements σ of G_n having a given signed descent set and such that σ^{-1} has a given signed descent set as a scalar product of two characters of G_n (Theorem 7).

In Section 4, Theorem 16 gives an equality between three symmetric functions. It has a relation with Poincaré–Birkhoff–Witt theorem and extends a result of [10]. See also [21]. This allows us to write the number of signed permutations having a given signed descent set and being in a fixed conjugacy class of G_n as a scalar product of two characters of G_n (Theorem 17).

We then apply these ideas to the study of Solomon descent sets in the hyperoctahedral groups B_n. We express the number of elements of B_n having a given descent set and being in a fixed conjugacy class of B_n as a scalar product of two characters of B_n (Corollary 24). Moreover, a scalar product of Solomon representations gives us the number of elements of B_n having given descent set and idown set (Corollary 25). These two results extend to the case of hyperoctahedral groups results of [8–10].

Finally, extending a result of Desarménien [4], we give some symmetric analogs of the exponential generating function of alternating elements of B_n. See also [25].

2. The characteristic function

Fix p a positive integer and ζ a primitive pth root of unity. For any $n\geqslant 0$, G_n denotes the wreath product of $\mathbb{Z}/p\mathbb{Z}$ with S_n; an element of G_n is of the form $(x_1,\dots,x_n,s)$ where $x_i\in\mathbb{Z}/p\mathbb{Z}$ and $s\in S_n$. If we denote $+$ the operation in $\mathbb{Z}/p\mathbb{Z}$, the multiplication rule in G_n is

$$(x_1',\dots,x_n',s')(x_1,\dots,x_n,s)=(x_1+x_{s(1)}',\dots,x_n+x_{s(n)}',s's).$$

We will consider that the underlying set of $\mathbb{Z}/p\mathbb{Z}$ is $\{1,\dots,p\}$ or $\{0,\dots,p-1\}$ (we identify 0 and p). We define the sign (resp. *absolute value*) of $(i,j)\in\{1,\dots,n\}\times\mathbb{Z}/p\mathbb{Z}$ as $\operatorname{sign}(i,j)=j$ (resp. $|(i,j)|=i$). This is easily seen that elements of G_n are permutations σ of $\{1,\dots,n\}\times\mathbb{Z}/p\mathbb{Z}$ such that

$$\sigma((i,j))=(|\sigma(i,0)|,\operatorname{sign}(\sigma(i,0))+j).$$

So each element $\sigma \in G_n$ is determined by the list $\sigma(1)\ldots\sigma(n)$, where i is identified with $(i,0)$. Moreover, $x_i = \operatorname{sign}(\sigma(i))$ and $s(i) = |\sigma(i)|$ if σ is denoted by $(x_1,\ldots,x_n,s)$; throughout this paper, we shall use these two notations depending on the context.

We denote $G = \bigcup_{n\geqslant 0} G_n$ (disjoint union), and the elements of G are called *signed permutations.* Let $\alpha = (x_1,\ldots,x_n,s)$ be in G_n. If $z = (i_1,\ldots,i_k)$ is a cycle of the permutation s then

$$\begin{aligned}
&((i_1,0),(i_2,x_{i_1}),\ldots,(i_k,x_{i_1}+\cdots+x_{i_{k-1}}),(i_1,x_{i_1}+\cdots+x_{i_k}),\ldots,(i_k,-x_{i_k})),\\
&((i_1,1),(i_2,x_{i_1}+1),\ldots,(i_1,x_{i_1}+\cdots+x_{i_k}+1),\ldots,(i_k,-x_{i_k}+1)),\\
&\ldots\\
&((i_1,p-1),(i_2,x_{i_1}+p-1),\ldots,(i_1,x_{i_1}+\cdots+x_{i_k}+p-1),\ldots,(i_k,-x_{i_k}+p-1))
\end{aligned}$$

are cycles of α viewed as a permutation of $\{1,\ldots,n\}\times\mathbb{Z}/p\mathbb{Z}$. Clearly, when we know one of these cycles, we know all the others, so we will denote this set of cycles

$$\begin{pmatrix} i_1 & \ldots & i_{k-1} & i_k \\ (i_2,x_{i_1}) & \ldots & (i_k,x_{i_{k-1}}) & (i_1,x_{i_k}) \end{pmatrix}.$$

Let $\mathscr{P}$ be the set of partitions. We denote by $\mathscr{P}^{G}$ the set of *partition valued functions on* $\mathbb{Z}/p\mathbb{Z}$, i.e., the set of functions from $\mathbb{Z}/p\mathbb{Z}$ to $\mathscr{P}$. We shall denote partition valued functions by bold Greek letters.

For each signed permutation $\alpha = (x_1,\ldots,x_n,s)$ we define a partition valued function, called the *cycle type*, $\operatorname{ct}(\alpha)$ on $\mathbb{Z}/p\mathbb{Z}$, such that the parts of $\operatorname{ct}(\alpha)(c)$, $c \in \mathbb{Z}/p\mathbb{Z}$, are the lengths of the cycles z in s such that $x_{i_1}+\cdots+x_{i_k} = c$.

The following proposition gives a classical result about conjugacy classes in G_n (see [13]).

Proposition 1. *Two signed permutations in G_n are in the same conjugacy class if and only if they have the same cycle type.* □

The *weight* (resp. *length*) $\|\boldsymbol{\lambda}\|$ (resp. $l(\boldsymbol{\lambda})$) of $\boldsymbol{\lambda}$ is the sum of the weights (resp. lengths) of the partitions $\boldsymbol{\lambda}(0),\boldsymbol{\lambda}(1),\ldots,\boldsymbol{\lambda}(p-1)$, where the *weight* (resp. *length*) of a partition is the sum (resp. number) of its parts. Observe that if α is element of G_n, the weight of $\operatorname{ct}(\alpha)$ is equal to n.

For any alphabet X, we denote by $\Lambda(X)$ the ring of symmetric functions, i.e. $\mathbb{Z}[e_1(X),e_2(X),\ldots]$. We also denote by $\Lambda_{\mathbb{C}}(X)$ the ring of symmetric functions on X with coefficients in $\mathbb{C}$, the field of complex numbers. If k is an integer, $p_k(X)$ denotes the power sum $\sum_{x\in X} x^k$. Let A be the disjoint union of p infinite alphabets $X^0,X^1,\ldots,X^{p-1}$. We define a mapping

$$\psi : G \to \Lambda_{\mathbb{C}}(X^0)\otimes\Lambda_{\mathbb{C}}(X^1)\otimes\cdots\otimes\Lambda_{\mathbb{C}}(X^{p-1}),$$

$$\psi(\alpha) = \prod_{i=0}^{p-1}\prod_{j=1}^{l(\boldsymbol{\lambda}(i))}(p_{\boldsymbol{\lambda}(i)_j}(X^0)+\zeta^i p_{\boldsymbol{\lambda}(i)_j}(X^1)+\cdots+\zeta^{i(p-1)}p_{\boldsymbol{\lambda}(i)_j}(X^{p-1})),$$

where λ equals $\mathrm{ct}(\alpha)$ and $\lambda(i)_j$ is the jth part of $\lambda(i)$. Clearly, ψ is a central function on G.

We are now able to define our essential tool, the *characteristic map* ch: for any character f of G_n, we write

$$\mathrm{ch}(f) = \langle f, \psi \rangle_{G_n} = \frac{1}{|G_n|} \sum_{\sigma \in G_n} f(\sigma)\psi(\sigma^{-1}).$$

Let R^n denote the $\mathbb{Z}$-module spanned by the irreducible characters of G_n, and let $R = \bigoplus_{n \geqslant 0} R^n$. If $f \in R^m$ and $g \in R^n$, then $f \times g$ is a character of $G_m \times G_n$. We embed $G_m \times G_n$ in G_{m+n} and we define

$$f \cdot g = (f \times g) \uparrow_{G_m \times G_n}^{G_{m+n}}.$$

This multiplication gives a ring structure on R. Let $f = \sum_{n \geqslant 0} f_n$ and $g = \sum_{n \geqslant 0} g_n$ with f_n and g_n elements of R^n. We define the scalar product $\langle f, g \rangle$ by

$$\langle f, g \rangle = \sum_{n \geqslant 0} \langle f_n, g_n \rangle_{G_n},$$

where $\langle -, - \rangle_{G_n}$ is the scalar product of central functions on G_n.

Define also an inner product on $\Lambda(X^0) \otimes \cdots \otimes \Lambda(X^{p-1})$ by $\langle f_0 \otimes \cdots \otimes f_{p-1}, g_0 \otimes \cdots \otimes g_{p-1} \rangle = \langle f_0, g_0 \rangle \cdots \langle f_{p-1}, g_{p-1} \rangle$, where $\langle f_i, g_i \rangle$ is the usual inner product on $\Lambda(X^i)$.

We define η_n^l the character of G_n which send each signed permutation of cycle type $\boldsymbol{\mu}$ to $\zeta^{l \sum_{i=0}^{p-1} i l(\mu(i))}$. Now, for any partition-valued function $\boldsymbol{\lambda}$, we define the virtual character $\chi^{\boldsymbol{\lambda}}$ by

$$\chi^{\boldsymbol{\lambda}} = \prod_{l=0}^{p-1} \det(\eta^l_{\lambda_i(l) - i + j})_{1 \leqslant i,j \leqslant n}.$$

We then have the following results.

Theorem 2. *The characteristic map is an isometric ring isomorphism of R onto $\Lambda(X^0) \otimes \cdots \otimes \Lambda(X^{p-1})$.*

Corollary 3. *The characters $\chi^{\boldsymbol{\lambda}}$ are the irreducible characters of G_n and the characteristic map sends $\chi^{\boldsymbol{\lambda}}$ of G_n onto products of Schur functions*

$$s_{\boldsymbol{\lambda}}(X^0, X^1, \ldots, X^{p-1}) = s_{\lambda(0)}(X^0) \ldots s_{\lambda(p-1)}(X^{p-1}).$$

We shall give just a sketch of the proof and some essential lemmas. This method is just a generalisation of the one of Macdonald in [16]. All the references to this book correspond to analogous results or proofs in the case $p = 1$. For a complete proof, the reader can see [19].

Let $X^0, \ldots, X^{p-1}, Y^0, \ldots, Y^{p-1}$ denote infinite sets of variables such that $X^i = \{x_1^i, x_2^i, \ldots\}$ and $Y^i = \{y_1^i, y_2^i, \ldots\}$. The notation X and Y is used instead of $(X^0, \ldots, X^{p-1})$ and $(Y^0, \ldots, Y^{p-1})$.

The following lemma gives a condition for duality of two bases of $\Lambda(X^0)\otimes\cdots\otimes\Lambda(X^{p-1})$.

Lemma 4. *If (u_λ) and (v_λ) are two bases of $\Lambda(X^0)\otimes\cdots\otimes\Lambda(X^{p-1})$, then the two following conditions are equivalent:*

(i) $\langle u_\lambda, v_\mu\rangle = \delta_{\lambda\mu}$,

(ii) $\sum_\lambda u_\lambda(X)v_\lambda(Y)=\prod_{i=0}^{p-1}\prod_{j,k\geqslant 1}(1-x_j^iy_k^i)^{-1}.$

Proof. To prove this lemma, it suffices to verify that [16, p. 33]

$$\sum_\lambda h_\lambda(X)m_\lambda(Y)=\prod_{i=0}^{p-1}\prod_{j,k\geqslant 1}(1-x_j^iy_k^i)^{-1}=\sum_\lambda u_\lambda(X)v_\lambda(Y),$$

and to make the same computation as in [16, p. 34]. □

Let $P_r^{\zeta^i}(X)=\sum_{j=0}^{p-1}\zeta^{ij}p_r(X^j)$, for all $r\in\mathbb{N}$ and $i\in\mathbb{Z}/p\mathbb{Z}$. If ρ is a partition with k parts, we define $P_\rho^{\zeta^i}(X)=P_{\rho_1}^{\zeta^i}(X)\ldots P_{\rho_k}^{\zeta^i}(X)$. For any partition-valued function $\boldsymbol{\lambda}$ we define

$$P_{\boldsymbol{\lambda}}(X)=P_{\lambda(0)}^{\zeta^0}(X)\ldots P_{\lambda(p-1)}^{\zeta^{p-1}}(X).$$

Now define $P(\zeta^i,t)$ and $H(\zeta^i,l,t)$ by

$$P(\zeta^i,t)=\sum_{r\geqslant 1}\frac{1}{p}P_r^{\zeta^i}(X)t^{r-1},$$

$$H(\zeta^i,l,t)=\prod_{k=0}^{p-1}\left(\sum_{r\geqslant 1}h_r(X^{k+l})t^r\right)^{\zeta^{ik}/p}$$

$$=\prod_{k=0}^{p-1}\prod_{j\geqslant 1}(1-x_j^{k+l}t)^{-\zeta^{ik}/p}\quad\text{(from [16, p.14]).}$$

We define $z_{\boldsymbol{\lambda}}$ by $\prod_{j=1}^{p-1}\prod_{i=1}^{n}\mathrm{i}^{n_i(j)}n_i(j)!$ and $\boldsymbol{\lambda}^-$ by $\boldsymbol{\lambda}^-(j)=\boldsymbol{\lambda}(-j)$, for all $j\in\mathbb{Z}/p\mathbb{Z}$.

The last equation of the following lemma gives a formula which allows us to compute the scalar product $\langle P_{\boldsymbol{\lambda}}, P_{\boldsymbol{\mu}}\rangle$. The others are steps of the proof of this result.

Lemma 5. *The following formulas hold*

(i) $P(\zeta^i,t)=\frac{\mathrm{d}}{\mathrm{d}t}\log H(\zeta^i,t)$ (from [16, p. 16]),

(ii) $\prod_{i=1}^{p-1}H(\zeta^i,l,t)=\prod_{j\geqslant 1}(1-x_j^lt)^{-1}=\sum_{n\geqslant 0}h_n(X^l)t^n$ (from [16, p. 14]),

(iii) $H(\zeta^i, 0, t) = \sum_{\lambda \in \mathscr{P}} z_\lambda^{-1} P_\lambda^{\zeta^i}(X) t^{|\lambda|} p^{-l(\lambda)}$ (from [16, p. 17]),

(iv) $\sum_\lambda z_\lambda^{-1} p^{-l(\lambda)} P_\lambda(X) P_\lambda(Y) = \prod_{l=0}^{p-1} \prod_{j,k} (1 - x_j^l y_k^{-l})^{-1}$ (from [16, p. 33]),

(v) $P_r^{\zeta^{-i}}(Y^0, Y^{p-1}, \ldots, Y^1) = P_r^{\zeta^i}(Y^0, Y^1, \ldots, Y^{p-1})$

(vi) $\sum_\lambda z_\lambda^{-1} p^{-l(\lambda)} P_\lambda(X) P_{\lambda^-}(Y) = \prod_{l=0}^{p-1} \prod_{j,k} (1 - x_j^l y_k^l)$ (from [16, p. 33]),

(vii) $\langle P_\lambda, P_\mu \rangle = z_\lambda p^{l(\lambda)} \delta_{\lambda\mu^-}$ (from [16, p. 35]).

Proof of Theorem 2 (*sketch*). One can see that $\psi(\alpha) = P_\lambda(X)$ if $\mathrm{ct}(\alpha) = \lambda$. Hence, if f_λ is the value of f at elements of cycle type λ, then one has

$$\mathrm{ch}(f) = \langle f, \psi \rangle = \sum_\lambda z_\lambda^{-1} p^{-l(\lambda)} f_\lambda P_{\lambda^-}(X) \quad \text{(from [16, p. 61])},$$

where the sum is over the partition valued functions of weight n.

Thus, by Lemma 5(vii) one has

$$\langle \mathrm{ch}(f), \mathrm{ch}(g) \rangle = \sum_\lambda z_\lambda^{-1} p^{-l(\lambda)} f_\lambda g_{\lambda^-} = \langle f, g \rangle \quad \text{(from [16, p. 61])}.$$

So ch is an isometric linear map of R to $\Lambda(X^0) \otimes \cdots \otimes \Lambda(X^{p-1})$.

In order to verify that ch is a ring homomorphism it suffices to note that, if α and β are elements of G_m and G_n respectively, then if we embed $G_m \times G_n$ in G_{m+n}, we have

$$\psi(\alpha \times \beta) = \psi(\alpha)\psi(\beta);$$

and to use the Frobenius reciprocity (see [16, p. 61] for a similar proof in the case $G_n = S_n$).

Now, using the relation (ii) of Lemma 5, one can easily see that $\mathrm{ch}(\eta_n^l) = h_n(X^l)$, so that ch is a bijective map of R onto $\Lambda(X^0) \otimes \cdots \otimes \Lambda(X^{p-1})$. □

Proof of Corollary 3. The fact that χ^λ is mapped onto

$$s_\lambda(X^0, X^1, \ldots, X^{p-1}) = s_{\lambda(0)}(X^0) \ldots s_{\lambda(p-1)}(X^{p-1})$$

comes trivially from the Jacobi–Trudi identity (see [16, p. 25]). From the classical case, one can also simply prove that χ^λ, and not $-\chi^\lambda$, is an irreducible character (it suffices to note, from the induction of characters, that $\chi^\lambda(1_{G_n}) \rangle 0$). □

We are now giving another essential tool in our study. This extends a part of Schur–Weyl duality to the context of wreath products.

Let A be the disjoint union of alphabets X^i. We shall denote by $\mathbb{C}\langle A \rangle$ the free associative algebra on A, and by $\mathbb{C}\langle A \rangle_n$ the subspace of $\mathbb{C}\langle A \rangle$ generated by the words of length n.

Fix n and assume that each X^i is equal to $X \times \{i\}$ where X is an alphabet containing $[n]$. Then A contains $[n] \times \mathbb{Z}/p\mathbb{Z}$. Denote by E_n the linear span of the words $w_\sigma = \sigma(1)\dots\sigma(n)$, for $\sigma \in G_n$.

The subspace E_n, viewed as a $\mathbb{C}$-vector space, may be identified with $\mathbb{C}[G_n]$ viewed as a $\mathbb{C}$-vector space.

We associate to each element τ of G_n an isomorphism $\underline{\tau}$ of $\mathbb{C}\langle A\rangle$ by

$$\begin{aligned} \underline{\tau}(k,j) &= \zeta^{j\cdot \mathrm{sign}(\tau(k,0))}(|\tau(k,0)|,j) && \text{if } k \in \{1,\dots,n\},\\ \underline{\tau}(a,j) &= (a,j) && \text{if } a \notin \{1,\dots,n\}. \end{aligned}$$

This defines a representation of G_n on E_n, called the *left action* of G_n.

Let $\alpha = (\alpha_a)_{a\in A}$ be some multi-index, with the α_a almost all zero. Denote by E_α the space of finely homogeneous polynomials of partial degree α_a in each letter a, and $|\alpha|$ the sum of the α_a.

Assume that F is a subspace of $\mathbb{C}\langle A\rangle_n$, invariant under each algebra isomorphism of $\mathbb{C}\langle A\rangle$ which sends each letter onto a linear combination of letters of the same sign. Then the space $F \cap E_n$, called the *multilinear part* of F, is invariant under the left action of G_n. Denote by χ_F the character of this action on $F \cap E_n$.

Theorem 6. *If* $n_\alpha = \dim(F \cap E_\alpha)$ *one has* $\mathrm{ch}(\chi_F) = \sum_{|\alpha|=n} n_\alpha \prod_{a\in A} a^{\alpha_a}$.

The reader can find a proof of this result in [19].

Example. If $p=2$ we use, respectively, X and $\bar{X}$ instead of X^0 and X^1. We assume that $n=2$ and $F = \langle\{[a,\bar{b}],[\bar{a},b] \mid a \leqslant b\}\rangle$, and we then have $E_2 \cap F = \langle\{[1,\bar{2}],[\bar{1},2]\}\rangle$. In order to compute the characteristic function of the representation on $E_2 \cap F$ we compute the characters for each conjugacy class in G_2.

$$\bar{1}2.[1,\bar{2}] = [1,\bar{2}],$$

$$\bar{1}2.[\bar{1},2] = -[\bar{1},2] \quad \text{hence } \chi_F(\bar{1}2) = 0,$$

$$\bar{1}\bar{2}.[1,\bar{2}] = -[1,\bar{2}],$$

$$\bar{1}\bar{2}.[\bar{1},2] = -[\bar{1},2] \quad \text{hence } \chi_F(\bar{1}\bar{2}) = -2,$$

$$21.[1,\bar{2}] = [2,\bar{1}],$$

$$21.[\bar{1},2] = [\bar{2},1] \quad \text{hence } \chi_F(21) = 0,$$

$$\bar{2}1.[1,\bar{2}] = [2,\bar{1}],$$

$$\bar{2}1.[\bar{1},2] = [1,\bar{2}] \quad \text{hence } \chi_F(\bar{2}1) = 0.$$

So, one has

$$\begin{aligned}\mathrm{ch}(\chi_F) &= \tfrac{1}{8}(1.2(p_1(X)+p_1(\bar{X}))^2+2.0(p_1(X)+p_1(\bar{X}))\\ &\quad\times(p_1(X)-p_1(\bar{X}))+1.(-2)(p_1(X)-p_1(\bar{X}))^2\\ &\quad+2.0(p_2(X)+p_2(\bar{X}))+2.0(p_2(X)-p_2(\bar{X})))\\ &= \tfrac{1}{8}(2(p_1(X)+p_1(\bar{X}))^2-2(p_1(X)-p_1(\bar{X}))^2)\\ &= p_1(X)p_1(\bar{X}).\end{aligned}$$

Recall that E_α is the space of finely homogeneous polynomials of partial degree α_a in each letter a and that $n_\alpha = E_\alpha \cap F$. One can easily see that n_α is nonzero if and only if there is one letter a in X and one letter $\bar{b}$ in $\bar{X}$ such that $\alpha(a)=1$ and $\alpha(\bar{b})=1$, and $\alpha(c)=0$ for all other letters c in A. Hence the generating function of F is equal to

$$\sum_{a,\bar{b}} a\bar{b} = p_1(X)p_1(\bar{X}) = \mathrm{ch}(\chi_F).$$

3. Signed permutations with given descent set and idown set

We denote $S(\sigma)$ the *descent set* of σ according to the right lexicographic order $<_{\mathrm{r}}$ on $\{1,\dots,n\}\times\mathbb{Z}/p\mathbb{Z}$: $i\in S(\sigma)\Leftrightarrow\sigma(i)<_{\mathrm{r}}\sigma(i+1)$.

The *sign function* $m(\sigma)$ of σ is defined by

$$\begin{aligned} m(\sigma):\{1,\dots,n\}&\to\mathbb{Z}/p\mathbb{Z}\\ i&\mapsto \mathrm{sign}(\sigma(i)).\end{aligned}$$

We also say that the *signed descent set* of σ is $\mathrm{sd}(\sigma)=[S(\sigma),m(\sigma)]$. For any $T\subseteq\{1,\dots,n-1\}$ and $m:\{1,\dots,n\}\to\mathbb{Z}/p\mathbb{Z}$. If $\leqslant_p$ is the order $1<_p 2<_p\cdots<_p p$ on $\mathbb{Z}/p\mathbb{Z}$, and if $i\notin T$ implies $m(i)\leqslant_p m(i+1)$, we say that $[T,m]$ is a *signed set* (this condition will permit us to have a bijection between signed sets and signed compositions).

We also say that $[T,m]$ is a *signed descent set* if there exists a signed permutation σ such that $\mathrm{sd}(\sigma)=[T,m]$. A *signed subset* $[T,m]$ of $\{1,\dots,n-1\}$ is a signed set such that T is a subset of $\{1,\dots,n-1\}$.

Suppose now that X is a totally ordered set. For any letter (a,i) in A we define the *absolute value*, $|(a,i)|=a$, and the sign, $\mathrm{sign}(a,i)=i$, of (a,i). We define the lexicographic order $\leqslant_1$ on $A=X\times\mathbb{Z}/p\mathbb{Z}$:

If $X=\{a<b<\cdots\}$ then

$$A=\{(a,1)<_1(a,2)<_1\cdots<_1(a,p)<_1(b,1)<_1(b,2)<_1\cdots\}.$$

In the following, we will often use the convention $(x,0)=(x,p)=x$.

Let $w=a_1\dots a_n$ be in A^*. The *standard signed permutation*, $\mathrm{sts}(w)$, (resp. *costandard signed permutation* $\mathrm{csts}(w)$) of w is the unique element β (resp. σ) of G_n

such that

$$\begin{cases} |\beta(i)| < |\beta(j)| \Leftrightarrow (a_i <_1 a_j \text{ or } (a_i = a_j \text{ and } i<j)) \\ \text{and} \\ \text{sign}(\beta(i)) = \text{sign}(a_i) \end{cases}$$

$$\left(\text{resp} \begin{cases} |\sigma(i)| < |\sigma(j)| \Leftrightarrow (a_i <_1 a_j \text{ or } (a_i = a_j \text{ and } i<j)) \\ \text{and} \\ \text{sign}(\sigma(i)) = -\text{sign}(a_i) \end{cases} \right).$$

Example. For $p=3$ if $w=(d,3)(c,1)(a,2)(b,3)(b,1)(a,3)(d,1)(a,3)$ then

$$\text{sts}(w) = (8,3)(6,1)(1,2)(5,3)(4,1)(2,3)(7,1)(3,3),$$
$$\text{csts}(w) = (8,3)(6,2)(1,1)(5,3)(4,2)(2,3)(7,2)(3,3).$$

The *evaluation* is the function

$$\text{ev} : A^* \to \mathbb{Z}[A]$$
$$w \mapsto w.$$

We use A at the same time as a set of noncommuting and as a set of commuting variables. This should cause no confusion.

We associate to each signed descent set $[T,m]$ the function of $\mathbb{Z}[[A]]$

$$S_{[T,m]} = \sum \text{ev}(w),$$

where the sum is over all the words w on A such that $[S(\text{sts}(w)), m(\text{sts}(w))] = [T,m]$. The discussion before Lemma 13 shows that $S_{[T,m]}$ is a product of skew Schur functions, so this is the characteristic of a certain representation $\chi_{[S,m]}$ of G_n.

For each $\sigma \in G_n$, denote $\tilde{\sigma}$ the element of G_n such that, for all $1 \leqslant i \leqslant n$, $\tilde{\sigma}(i) = (j,-k)$ if $\sigma(i) = (j,k)$. Observe that if $\text{sts}(w) = \sigma$ then $\text{csts}(w) = \tilde{\sigma}$, moreover, this operation commutes with the one which sends σ to σ^{-1}, so there is no ambiguity in the notation $\tilde{\sigma}^{-1}$.

The following result extends a theorem of Gessel [9]. This is the principal result of the section.

Theorem 7. *The number of σ in G_n with signed descent set $[T,m]$ and such that the signed descent set of $\tilde{\sigma}^{-1}$ is $[T',m']$ is $\langle \chi_{[T,m]}, \chi_{[T',m']} \rangle_{G_n}$.*

To prove this theorem we need some definitions and lemmas.

First we define *signed compositions* (C,v) of n by

— C is a composition $(c_1,\ldots,c_k)$ of n.

— v is a matrix of natural integers with k rows and p columns with $\sum_{j=1}^{p} v_{i,j} = c_i$.

This notation is redundant, because the information about the composition C is contained in the matrix v, but it is convenient. There exists a one-to-one map, *set*, between signed compositions of n and signed subsets of n, defined by: $[T,m]=\text{set}(C,v)$ if and only if $T=\{c_1,c_1+c_2,\dots,c_1+\cdots+c_{k-1}\}$ and there are $v_{s,j}$ i in $\{c_1+\cdots+c_{s-1}+1,\dots,c_1+\cdots+c_s\}$ such that $m(i)=j$. Because of the condition $i\notin T \Rightarrow m(i)\leqslant_p m(i+1)$, one has elements of sign 1, then elements of sign 2... The inverse of *set* will be denoted by *comp*.

Because of this bijection, we shall say that objects indexed by signed sets are also indexed by signed compositions. For example, if $\text{comp}(C,v)=[T,m]$ we will write $S_{(C,v)}=S_{[T,m]}$.

To each signed permutation σ one can associate the *signed descent composition of* σ $\text{CS}(\sigma)=(C(\sigma),v(\sigma))$ such that $\text{set}(\text{CS}(\sigma))=[S(\sigma),m(\sigma)]$. The signed compositions which are signed descent composition of some signed permutations are called *descent signed compositions*. Clearly, the function *set* is a bijection from the set of signed descent compositions of n onto the set of signed descent subsets of $\{1,\dots,n-1\}$.

Define a partial order on the signed sets by

$$[S,m]\leqslant[S',m'] \Leftrightarrow \begin{cases} S\subseteq S', \\ m=m'. \end{cases}$$

We have a corresponding order on signed compositions induced by 'set' and defined by

$$(C,u)\leqslant(D,v) \Leftrightarrow \text{set}(C,u)\leqslant\text{set}(D,v).$$

Now, define *monomial quasi-symmetric functions*, if $\text{set}(C,v)=[T,m]$ is a signed subset of $\{1,\dots,n-1\}$:

$$\begin{aligned} M_{(C,v)} &= \sum_{\substack{t_1<\cdots<t_k\\ t_i\in X}} (t_1,1)^{v_{1,1}}(t_1,2)^{v_{1,2}}\dots(t_1,p)^{v_{1,p}}(t_2,1)^{v_{2,1}}\dots(t_k,p)^{v_{k,p}} \\ &= M_{[T,m]}=\sum_{\substack{t_i=t_{i+1}\text{ if } i\notin T\\ t_i<t_{i+1}\text{ if } i\in T\\ t_i\in X}} (t_1,m(1))\dots(t_n,m(n)). \end{aligned} \tag{1}$$

For all signed subsets $[T,m]$ of $\{1,\dots,n-1\}$ we define the formal series

$$F_{[T,m]}=\sum_{x_i\leqslant_1 x_{i+1}} x_1\dots x_n, \tag{2}$$

where each x_i must have the sign $m(i)$, and $x_i<_1 x_{i+1}$ if $i\in T$.

We then have the following lemma.

Lemma 8. *For any signed set* $[T,m]$ *one has* $F_{[T,m]}=\sum_{[T,m]\leqslant[T',m']}M_{[T',m']}$.

Proof. This is obvious from (2) and (1). □

Example. We take $p=2$, $n=5$ and for any $a\in X$, a stands for $(a,0)$ and $\bar{a}$ for $(a,1)$; moreover, we write m as a list of $+$ and $-$: if $m(i)=0$ we put a $+$ at the ith place of the list and if $m(i)=1$ we put a $-$. We then have

$$\begin{aligned}F_{[\{2,3\},-+-++]} &= \sum_{a\leqslant b<c<d\leqslant e} \bar{a}b\bar{c}de = \sum_{a<b<c} \bar{a}a\bar{c}d^2 + \sum_{a<b<c<d} \bar{a}a\bar{b}cd \\ &\quad + \sum_{a<b<c<d} \bar{a}b\bar{c}d^2 + \sum_{a<b<c<d<e} \bar{a}b\bar{c}de \\ &= M_{[\{2,3\},-+-++]} + M_{[\{2,3,4\},-+-++]} + M_{[\{1,2,3\},-+-++]} \\ &\quad + M_{[\{1,2,3,4\},-+-++]}.\end{aligned}$$

Corollary 9. *The $F_{[S,m]}$ are linearly independent.*

Proof. The $M_{[S,m]}$ are linearly independent and one can invert the relation of Lemma 8. □

Lemma 10. *For any signed permutation σ one has*

$$\operatorname{sign}(\sigma|\sigma^{-1}(i)|) = -\operatorname{sign}(\sigma^{-1}(i)) \quad \textit{for all } i \in [n].$$

Proof. Let $i \in [n]$, so

$$\begin{aligned}\sigma^{-1}(i)=(k,j) &\Leftrightarrow \sigma(k,j)=(i,0) \\ &\Leftrightarrow \sigma(k)=(i,-j)=\sigma(|\sigma^{-1}(i)|). \qquad \square\end{aligned}$$

The following result gives an expression of the $F_{(C,v)}$ in term of evaluation of some words. This relation is essential in studying the links between quasi-symmetric functions and symmetric functions.

Lemma 11. *For any signed permutation σ one has*

$$F_{\mathrm{CS}(\sigma^{-1})} = \sum_{\mathrm{csts}(x_1\ldots x_n)=\sigma} \mathrm{ev}(x_1\ldots x_n).$$

Proof. Denote $(C,v)=\mathrm{CS}(\sigma^{-1})$ and $[S,m]=\mathrm{sd}(\sigma^{-1})=\mathrm{set}(C,v)$. In order to simplify the notation, for each element τ of G_n, we shall write $\tau(k)$ instead of $\tau((k,0))$. First note that:

- $k \notin S \Leftrightarrow \sigma^{-1}(k) <_r \sigma^{-1}(k+1)$

$$\Leftrightarrow \left| \begin{array}{l} |\sigma^{-1}(k)| < |\sigma^{-1}(k+1)| \text{ and } \operatorname{sign}\sigma^{-1}(k) \leqslant_p \operatorname{sign}\sigma^{-1}(k+1) \\ \text{or} \\ \operatorname{sign}\sigma^{-1}(k) <_p \operatorname{sign}\sigma^{-1}(k+1) \end{array} \right. \tag{3}$$

$$\Leftrightarrow \left| \begin{array}{l} (k \text{ appears to the left of } k+1 \text{ in } |\sigma(1)|\ldots|\sigma(n)| \text{ and} \\ \operatorname{sign}\sigma^{-1}(k) \leqslant_p \operatorname{sign}\sigma^{-1}(k+1)). \\ \text{or} \\ \operatorname{sign}\sigma^{-1}(k) <_p \operatorname{sign}\sigma^{-1}(k+1). \end{array} \right. \tag{4}$$

- Recall that $\sigma = \mathrm{csts}(x_1 \ldots x_n)$ is equivalent to

$$\begin{cases} |\sigma(i)| < |\sigma(j)| \Leftrightarrow (x_i <_1 x_j \text{ or } (x_i = x_j \text{ and } i < j)) \\ \text{and} \\ \mathrm{sign}(\sigma(i)) = -\,\mathrm{sign}(x_i). \end{cases} \tag{5}$$

Now, by Eq. (2) $F_{(C,v)} = \sum t_1 \ldots t_n$ with $|t_i| \leqslant |t_{i+1}|$, $|t_i| < |t_{i+1}|$ if $\sigma^{-1}(i+1) <_r \sigma^{-1}(i)$ and $\mathrm{sign}(t_i) = \mathrm{sign}(\sigma^{-1}(i))$.

If we set $x_i = t_{|\sigma(i)|}$ and $\alpha(i) = |\sigma^{-1}(i)|$, we obtain $F_{(C,v)} = \sum \mathrm{ev}(x_1 \ldots x_n)$ with the following conditions on the x_i (since $t_i = x_{|\sigma^{-1}(i)|} = x_{\alpha(i)}$):

$$\begin{cases} |x_{\alpha(i)}| \leqslant |x_{\alpha(i+1)}|, \\ |x_{\alpha(i)}| < |x_{\alpha(i+1)}| \quad \text{if } \sigma^{-1}(i+1) <_r \sigma^{-1}(i), \\ \mathrm{sign}(x_{\alpha(i)}) = \mathrm{sign}(\sigma^{-1}(i)). \end{cases} \tag{6}$$

We have to prove that conditions (6) and $\sigma = \mathrm{csts}(x_1 \ldots x_n)$ are equivalent.

By Lemma 10, the last condition in (6) is equivalent to

$$\mathrm{sign}(x_{\alpha(i)}) = \mathrm{sign}(\sigma^{-1}(i)) = -\,\mathrm{sign}(\sigma(|\sigma^{-1}(i)|)).$$

This is the last condition in (5) if we replace i by $|\sigma^{-1}(i)|$. Thus, the last conditions in (5) and in (6) are equivalent.

Assuming that (6) is verified, we just have to show that $|\sigma(i)| < |\sigma(j)|$ implies ($x_i <_1 x_j$ or ($x_i = x_j$ and $i < j$) ($\diamond$)), because the two conditions give total orders on $[n]$.

If $|\sigma(i)| < |\sigma(j)|$, then by the first assertion of (6) one has

$$|x_i| = |x_{|\alpha|\sigma(i)||}| \leqslant |x_{|\alpha(|\sigma(i)|+1)|}| \leqslant \cdots \leqslant |x_{|\alpha|\sigma(j)||}| = |x_j| \tag{7}$$

If $|x_i| < |x_j|$, then $x_i <_1 x_j$ and ($\diamond$) holds.

If $|x_i| = |x_j| = a$, then for all $k \in \{|\sigma(i)|, \ldots, |\sigma(j)| - 1\}$ one has, by (7), $|x_i| \leqslant |x_{\alpha(k)}| \leqslant |x_j|$ so $|x_{\alpha(k)}| = |x_{\alpha(k+1)}|$. Now, the second condition in equation (6) implies $S \cap \{|\sigma(i)|, |\sigma(i)| + 1, \ldots, |\sigma(j)| - 1\} = \emptyset$, and the equivalence (3) shows that for all $k \in \{|\sigma(i)|, \ldots, |\sigma(j)| - 1\}$ one has

$$\mathrm{sign}(\sigma^{-1}(k)) \leqslant_p \mathrm{sign}(\sigma^{-1}(k+1)). \tag{8}$$

We now have to study the three following subcases:

— If $\mathrm{sign}(x_i) >_p \mathrm{sign}(x_j)$, then there exists a $k \in \{|\sigma(i)|, \ldots, |\sigma(j)| - 1\}$ such that $\mathrm{sign}(\sigma^{-1}(k)) >_p \mathrm{sign}(\sigma^{-1}(k+1))$, which is impossible by (8).

— If $\mathrm{sign}(x_i) <_p \mathrm{sign}(x_j)$, then $x_i < x_j$ and ($\diamond$) holds.

— If $\mathrm{sign}(x_i) = \mathrm{sign}(x_j) = s$, then for all k in $\{|\sigma(i)|, \ldots, |\sigma(j)| - 1\}$ one has $\mathrm{sign}(\sigma^{-1}$ $(k) = s$. But in this case we cannot have $\mathrm{sign}(\sigma^{-1}(k)) <_p \mathrm{sign}(\sigma^{-1}(k+1))$. So by virtue of (4), k appears to the left of $k+1$ in $|\sigma(1)| \ldots |\sigma(n)|$, and $|\sigma(i)|$ appears to the left of $|\sigma(j)|$; thus $i < j$ and ($\diamond$) holds.

On the other hand, suppose that $\sigma = \mathrm{csts}(x_1 \ldots x_n)$, hence, by (5),

$$|\sigma(i)| < |\sigma(j)| \Leftrightarrow (x_i <_1 x_j \text{ or } (x_i = x_j \text{ and } i < j)).$$

Because $|\sigma|\sigma^{-1}(i)||=i<i+1=|\sigma|\sigma^{-1}(i+1)||$, one has

$$x_{|\sigma^{-1}(i)|}<_1 x_{|\sigma^{-1}(i+1)|} \quad \text{or} \quad (x_{|\sigma^{-1}(i)|}=x_{|\sigma^{-1}(i+1)|} \quad \text{and} \quad |\sigma^{-1}(i)|<|\sigma^{-1}(i+1)|). \tag{$*$}$$

The first part of $(*)$ is equivalent to (recall that $\alpha(i)=|\sigma^{-1}(i)|$)

$$|x_{\alpha(i)}|<|x_{\alpha(i+1)}| \quad \text{or} \quad \begin{cases} |x_{\alpha(i)}|=|x_{\alpha(i+1)}| \\ \operatorname{sign}(x_{\alpha(i)})<_p \operatorname{sign}(x_{\alpha(i+1)}). \end{cases}$$

The second part of $(*)$ is equivalent to

$$(|x_{\alpha(i)}|=|x_{\alpha(i+1)}| \text{ and } \operatorname{sign} x_{\alpha(i)}=\operatorname{sign} x_{\alpha(i+1)} \text{ and } |\sigma^{-1}(i)|<|\sigma^{-1}(i+1)|).$$

The equivalence of the last conditions in (5) and (6) gives us $\operatorname{sign} x_{\alpha(i)}=\operatorname{sign}\sigma^{-1}(i)$, so $(*)$ is equivalent to

$$|x_{|\sigma^{-1}(i)|}|<|x_{|\sigma^{-1}(i+1)|}| \quad \text{or} \quad \begin{cases} |x_{|\sigma^{-1}(i)|}|=|x_{|\sigma^{-1}(i+1)|}| \\ \text{and} \\ \left| \begin{array}{l} \begin{cases} \operatorname{sign}\sigma^{-1}(i)=\operatorname{sign}\sigma^{-1}(i+1) \\ \text{and} \\ |\sigma^{-1}(i)|<|\sigma^{-1}(i+1)| \end{cases} \\ \text{or} \\ \operatorname{sign}\sigma^{-1}(i)<_p \operatorname{sign}\sigma^{-1}(i+1). \end{array} \right. \end{cases}$$

By (3) we deduce (6). □

We now define *monomial* and *complete* symmetric functions, indexed by functions from $\mathbb{Z}/p\mathbb{Z}$ to the set of partitions (the *partition-valued functions*), by $m_\lambda=m_{\lambda(1)}(X^1)\dots m_{\lambda(p)}(X^p)$ and $h_\lambda=h_{\lambda(1)}(X^1)\dots h_{\lambda(p)}(X^p)$, where $m_{\lambda(i)}(X^i)$ (resp. $h_{\lambda(i)}(X^i)$) is the classical monomial (resp. complete) function on the set of variables X^i. See [16].

We say that a signed composition (C,v) is *compatible* with a partition-valued function λ and we write $\lambda=\lambda(C,v)$ if for each i, $\lambda(i)$ is a reordering of $(v_{1,i},v_{2,i},\dots,v_{k,i})$ (the zeros are omitted). By a straightforward calculation one has the following result:

Lemma 12. $m_\lambda(X^1,X^2,\dots,X^p)=\sum_{\lambda(C,v)=\lambda} M_{(C,v)}$.

Example. If $p=2$, $\lambda(1)=(1)$ and $\lambda(0)=(2,1)$ then

$$m_\lambda=m_1(X^1)m_{2,1}(X^0)=\left(\sum_t \bar t\right)\left(\sum_{x<y} x^2y+y^2x\right)$$

$$=\sum_{t<x<y}(\bar t x^2y+\bar t xy^2)+\sum_{x<y}(\bar x x^2y+\bar x xy^2)+\sum_{x<t<y}(x^2\bar t y+x\bar t y^2)$$

$$+\sum_{x<y}(x^2\bar{y}y+x\bar{y}y^2)+\sum_{x<y<t}(x^2y\bar{t}+xy^2\bar{t})$$

$$=M_{\substack{10\\02\\01}}+M_{\substack{10\\01\\02}}+M_{\substack{12\\01}}+M_{\substack{11\\02}}+M_{\substack{02\\10\\01}}+M_{\substack{01\\10\\02}}$$

$$+M_{\substack{02\\11}}+M_{\substack{01\\12}}+M_{\substack{02\\01\\10}}+M_{\substack{01\\02\\10}}.$$

For any signed composition (C,v), we denote by $m_{(C,v)}$ (resp. $h_{(C,v)}$) the monomial (resp. complete) function $m_{\lambda(C,v)}$ (resp. $h_{\lambda(C,v)}$).

Now, we need the *filling order* $<_{\mathrm{f}}$ on A, according to the order $a<b\dots$ on X;

$$A=\{(a,1)<_{\mathrm{f}}(b,1)<_{\mathrm{f}}\cdots<_{\mathrm{f}}(a,2)<_{\mathrm{f}}\cdots<_{\mathrm{f}}(a,p)<_{\mathrm{f}}(b,p)\cdots\}.$$

A *p-skew diagram* is a skew diagram where each row is composed of p blocks, possibly of size zero. If ρ is a p-skew diagram, a *p-semi-standard tableau* of shape ρ is an array obtained by replacing the dots of ρ with elements of A, such that the rows weakly increase and the columns strictly increase (from bottom to top) according to the filling order, and such that in each row the ith block is filled by letters in X^i, for each $i\in\mathbb{Z}/p\mathbb{Z}$.

Example. $(a,2)(d,3)$ is a 3-semi-standard tableau for the 3-skew

$$\begin{array}{lll}(a,2)(d,3) & & \\ & (c,1)(c,3)(b,3) & \\ & & (b,1)(a,2)(a,2)\end{array}$$

diagram

$$\begin{array}{lll}|\bullet|\bullet & & \\ & \bullet||\bullet\,\bullet & \\ & & \bullet|\bullet\,\bullet|.\end{array}$$

Remark. For some p-skew diagrams ρ there is no p-semi-standard tableau of shape ρ. For example, $\begin{array}{l} \;\;\bullet|\bullet \\ |\bullet\;\;\bullet \end{array}$ cannot be filled so that the top left letter is greater than the bottom left.

To each signed composition (C,v), one associates a *p-rim hook*, $\mathrm{rh}(C,v)$, (resp. a *p-horizontal strip*, $\mathrm{hs}(C,v)$), by

— the ith row of $\mathrm{rh}(C,v)$ and $\mathrm{hs}(C,v)$ has c_i dots.

— the kth block in the ith row of $\mathrm{rh}(C,v)$ and $\mathrm{hs}(C,v)$ has $v_{i,k}$ dots.

— the last dot of the ith row is over (resp. over and left) the first one of the $(i+1)$th row in $\mathrm{rh}(C,v)$ (resp. in $\mathrm{hs}(C,v)$).

Here we enumerate the rows from top to bottom.

Example. For

$$(C,v)=\left((4,5,3),\begin{pmatrix}2&0&1&1\\1&2&0&2\\0&3&0&0\end{pmatrix}\right)$$

$$\mathrm{rh}(C,v)=\begin{array}{lll}\bullet\bullet\,\|\,\bullet\,|\,\bullet & & \\ & \bullet|\,\bullet\bullet\|\bullet\;\bullet & \\ & & |\bullet\;\bullet\bullet\,\|\end{array}\quad\text{and}\quad \mathrm{hs}(C,v)=\begin{array}{lll}\bullet\bullet\,\|\,\bullet\,|\,\bullet & & \\ & \bullet|\,\bullet\bullet\|\,\bullet\bullet & \\ & & |\bullet\bullet\bullet\,\|\end{array}$$

The *evaluation*, $\mathrm{ev}(T)$, of a p-semi-standard tableau T is the commutative product, with multiplicities, of the elements of A appearing in T.

It is now easy to see that $S_{(C,v)}=S_{\mathrm{set}(C,v)}=\sum_{\mathrm{CS}(\mathrm{sts}(w))=(C,v)}\mathrm{ev}(w)$ (resp. $h_{(C,v)}$) is the sum of the evaluations of the p-semi-standard tableaux of shape $\mathrm{rh}(C,v)$ (resp. $\mathrm{hs}(C,v)$). This shows that $S_{(C,v)}$ is a product of skew Schur functions, so this is the characteristic of a representation of G_n (see [16]).

One can easily deduce the following lemma:

Lemma 13. *For any signed composition (C,v) one has*

$$h_{(C,v)}=\sum_{(D,u)\leqslant(C,v)}S_{(D,u)}.$$

Example. If

$$(C,v)=\left((3,1,2),\begin{pmatrix}2&1\\0&1\\2&0\end{pmatrix}\right)$$

then $\mathrm{set}(C,v)=[\{3,4\},--++--]$ (here if $m(i)=0$ we put a $+$ in the list and if $m(i)=1$ we put a $-$), so

$$[S,m]\leqslant\mathrm{set}(C,v)\;\Leftrightarrow\;[S,m]\in\{[\{3,4\},--++--],\ [\{4\},--++--]\}$$

thus

$$(D,u)\leqslant(C,v)\;\Leftrightarrow\;(D,u)\in\left\{\left((3,1,2),\begin{pmatrix}2&1\\0&1\\2&0\end{pmatrix}\right);\left((4,2),\begin{pmatrix}2&2\\2&0\end{pmatrix}\right)\right\}$$

so that

$$\sum_{(D,u)\leqslant(C,v)}S_{(D,u)}=\sum_{\substack{a\leqslant b\\ d<c\\ e\leqslant f}}\bar a\bar b cd\bar e\bar f+\sum_{\substack{a\leqslant b\\ c\leqslant d\\ e\leqslant f}}\bar a\bar b cd\bar e\bar f=\sum_{\substack{a\leqslant b\\ c,d\\ e\leqslant f}}\bar a\bar b cd\bar e\bar f$$

$$=h_2(\bar X)h_1(X)h_1(X)h_2(\bar X)=h_{(C,v)}.$$

One has the following lemma, extending Gessel's theorem in [9].

Lemma 14. *Let g be an element of $\Lambda(X^0)\otimes\cdots\otimes\Lambda(X^{p-1})$. Then*

$$g=\sum_{(C,v)}\langle g,S_{(C,v)}\rangle F_{(C,v)}.$$

Proof. The bases $(h_\rho)_{\rho\in\mathscr{P}}$ and $(m_\rho)_{\rho\in\mathscr{P}}$ are dual, so the bases h_λ and m_λ are dual bases of $\Lambda(X^1)\otimes\cdots\otimes\Lambda(X^p)$ and we have

$$\begin{aligned}
g &= \sum_{\rho}\langle g,h_\rho\rangle m_\rho \\
&= \sum_{\rho}\langle g,h_\rho\rangle \sum_{\lambda(D,u)=\rho} M_{(D,u)} && \text{by Lemma 12} \\
&= \sum_{(D,u)}\langle g,h_{(D,u)}\rangle M_{(D,u)} \\
&= \sum_{(D,u)}\left\langle g,\sum_{(C,v)\leqslant(D,u)} S_{(C,v)}\right\rangle M_{(D,u)} && \text{by Lemma 13} \\
&= \sum_{(C,v)\leqslant(D,u)}\langle g,S_{(C,v)}\rangle M_{(D,u)} \\
&= \sum_{(C,v)}\langle g,S_{(C,v)}\rangle \sum_{(C,v)\leqslant(D,u)} M_{(D,u)} \\
&= \sum_{(C,v)}\langle g,S_{(C,v)}\rangle F_{(C,v)} && \text{by Lemma 8.} \quad \square
\end{aligned}$$

If $\Pi\subseteq\bigcup_{n\geqslant 0}G_n$, we define the *quasi-symmetric generating function* of Π to be the series

$$\sum_{\sigma\in\Pi}F_{[S(\sigma),m(\sigma)]}.$$

So we can state the following lemma.

Lemma 15. *If $\Pi\in\bigcup_{n\geqslant 0}G_n$ and if the quasi-symmetric generating function g of Π is symmetric in each alphabet X^i, then the number of elements of Π which have signed descent composition (C,v) is $\langle g,S_{(C,v)}\rangle$.*

Proof. Let the number considered be $\alpha_{(C,v)}$. Then, by Lemma 14,

$$g=\sum_{(C,v)}\alpha_{(C,v)}F_{(C,v)}=\sum_{(C,v)}\langle g,S_{(C,v)}\rangle F_{(C,v)}$$

and $\alpha_{(C,v)}=\langle g,S_{(C,v)}\rangle$ since the $F_{(C,v)}$ are linearly independent. $\square$

Proof of Theorem 7. Recall that $\tilde\sigma(i)=(k,-j)$ if $\sigma(i)=(k,j)$. So we have

$$\begin{aligned}
S_{(C,v)} &= \sum_{\mathrm{CS}(\mathrm{sts}(w))=(C,v)} \mathrm{ev}(w) \\
&= \sum_{\mathrm{CS}(\sigma)=(C,v)}\ \sum_{\mathrm{sts}(w)=\sigma} \mathrm{ev}(w)
\end{aligned}$$

$$= \sum_{\mathrm{CS}(\sigma)=(C,v)} \sum_{\mathrm{csts}(w)=\tilde{\sigma}} \mathrm{ev}(w)$$
$$= \sum_{\mathrm{CS}(\sigma)=(C,v)} F_{\mathrm{CS}(\tilde{\sigma}^{-1})} \quad \text{by Lemma 11.}$$

But if $\tilde{\sigma}^{-1}=\beta$ then $\sigma=\tilde{\beta}^{-1}$, because the operation $\sigma \mapsto \tilde{\sigma}$ is an involution. Hence, $S_{(C,v)}$ is the quasi-symmetric generating function of the elements σ in $\bigcup_{n\geqslant 0} G_n$ such that the signed descent composition of $\tilde{\sigma}^{-1}$ is (C,v).

Theorem 7 is now a consequence of Lemma 15. □

4. Signed permutations with given cycle type and descent set

In this section we give a formula for the numbers of signed permutations with a given signed descent set and which are in a given conjugacy class. Such as in the preceding section, this formula is based on a scalar product of representations of G_n.

Denote by $\mathscr{L}(A)$ the free Lie algebra on A, i.e., the smallest submodule of $\mathbb{C}\langle A\rangle$, the free associative $\mathbb{C}$-algebra on A, containing A and closed under Lie bracket. The *Lie polynomials* are the elements of $\mathscr{L}(A)$. A *homogeneous Lie polynomial* is an element of $\mathscr{L}(A)$ which is a linear combination of words of the same length. Define *bar* to be the algebra endomorphism of $\mathbb{C}\langle A\rangle$ such that $bar(a)=\zeta^i a$ if $a\in X^i$. The *symmetrized product* of k elements $P_1,\dots,P_k$ of $\mathbb{C}\langle A\rangle$ is defined by

$$(P_1,\dots,P_k)=\frac{1}{k!}\sum_{\alpha\in S_k} P_{\alpha(1)}\dots P_{\alpha(k)}.$$

If λ is a partition-valued function on $\mathbb{Z}/p\mathbb{Z}$, we define U_λ to be the subspace of $\mathbb{C}\langle A\rangle$ linearly generated by the elements

$$(P_{0_1},\dots,P_{0_{l(\lambda(0))}},P_{1_1},\dots,P_{(p-1)_1},\dots,P_{(p-1)_{l(\lambda(p-1))}}),$$

where each P_{j_s} is a homogeneous Lie polynomial of degree $\lambda_s(j)$ with $\lambda(j)=\lambda_1(j)\dots\lambda_{l(\lambda(j))}(j)$ and $\mathrm{bar}(P_{j_s})=\zeta^j P_{j_s}$. The latter condition means that for each monomial $w=b_1\dots b_n$ appearing in P_{j_s}, the sum of the signs of the letters $b_1,\dots,b_n$ is equal to j.

The space $\mathscr{L}(A)$ is invariant under the action of each algebra endomorphism of $\mathbb{C}\langle A\rangle$ sending each letter $a\in X^i$ $(0\leqslant i\leqslant p-1)$ onto a linear combination of letters of X^i. So U_λ has the same property and we will be able to apply Theorem 6 to it.

A *necklace* of length n, or a *circular word*, is an n-gon in an oriented plane whose vertices are in A. It can be identified with the subset of A^* whose elements are the words of length n, constructed by running through the n-gon, starting from each vertice (for more information see [15], Section 1.3.).

A *primitive necklace* is a necklace of length n such that the associated set has exactly n elements.

The *evaluation* of a necklace is the monomial composed of the letters of the necklace considered as commutative variables. The *evaluation* (resp.length) of a multi-set of necklaces is the product (resp.sum) of the evaluations (resp.lengths) of its components, with multiplicities.

If L is a set of multi-sets of necklaces, the *generating function* of L is the sum of the evaluations of the elements of L.

The sign of a necklace is the sum, with multiplicities, of the signs of the letters in the necklace. If λ is a partition valued-function on $\mathbb{Z}/p\mathbb{Z}$ such that $\lambda(j) = 1^{n_1(j)}2^{n_2(j)}\ldots$ for $0 \leqslant j \leqslant p-1$, one has the following result.

Theorem 16. *The generating function* T_λ *of* $U_\lambda \cap \mathbb{C}\langle A\rangle_n$, *is equal to each of the following function*:
(i) *the generating function* PN_λ *of the multi-sets of primitive necklaces, with* $n_i(j)$ *necklaces of length* i *and sign* j, *for* $0 \leqslant j \leqslant p-1$.
(ii) *the quasisymmetric generating function of the signed permutations of cycle type* λ.
(iii) *the characteristic of the representation of* G_n *on the multilinear part of* U_λ.

The *generating function* of U_λ, $T_\lambda = \sum_{|\alpha|=\|\lambda\|} \dim(E_\alpha \cap U_\lambda) \prod_{b\in A} b^{\alpha_b}$ is symmetric in each set of variables X^i, because of the invariance of U_λ under the exchange of letters of X^i. Theorem 6 shows that T_λ is the characteristic of a certain character χ_λ, and we have the following theorem, extending result of Gessel and Reutenauer [10].

Theorem 17. *The number of elements* σ *in* G_n, *of cycle type* λ *and with signed descent set* $[S,m]$ *is* $\langle \chi_\lambda, \chi_{[S,m]}\rangle$.

Proof. This is clear from Lemma 15 and Theorem 16. □

Proof of Theorem 16. First, a *Lyndon word* in A^* is a nonempty word which is smaller than all its proper right factors for the dictionary order. The evaluation of a multi-set of Lyndon words is the product of the evaluations of each word in the multi-set.

We can now recall the following result (see, e.g., [22, p. 166]).

There are evaluation-preserving bijections between the three following sets:
(i) the set of words of length n in A^*;
(ii) the set of multi-sets of length n of Lyndon words in A^*;
(iii) the set of multi-sets of length n of primitive necklaces.

We use the bijection between (i) and (iii) given by Gessel and Reutenauer [10]. It uses the *standard permutation* $\mathrm{st}(w)$ of $w = b_1 \ldots b_n$, where $\mathrm{st}(w)$ is an element of S_n. This is the permutation such that $\mathrm{st}(w)(i) = |\mathrm{sts}(w)(i)| = |\mathrm{csts}(w)(i)|$ for $i \in \{1,\ldots,n\}$.

In the proof of the preceding result, Gessel and Reutenauer show that one has a bijection between words which have standard permutation of cycle type μ and multi-sets of primitive necklaces with m_i necklaces of length i ($\mu = 1^{m_1}2^{m_2}\ldots$).

Let $w=b_1\dots b_n$; to each cycle $(i_1,\dots,i_k)$ of $\mathrm{st}(w)$ corresponds a primitive necklace $\{b_{i_1}b_{i_2}\dots b_{i_k}, b_{i_2}\dots b_{i_k}b_{i_1},\dots\}$. So that $\mathrm{sts}(w)$ has some cycles of the form

$$\begin{pmatrix} i_1 & \cdots & i_{k-1} & i_k \\ (i_2,\varepsilon_{i_1}) & \cdots & (i_k,\varepsilon_{i_{k-1}}) & (i_1,\varepsilon_{i_k}) \end{pmatrix},$$

where $\varepsilon_{i_j}=\mathrm{sign}(b_{i_j})$.

We say that the *sign* of this cycle is $j=\varepsilon_{i_1}+\cdots+\varepsilon_{i_k}$, so if λ is the cycle type of $\mathrm{sts}(w)$, this cycle counts for one part of length k in $\lambda(j)$.

We suppose, until the end of this proof, that $\lambda(j)=1^{n_1(j)}2^{n_2(j)}\dots$, for $0\leqslant j\leqslant p-1$. The sign of the necklace $\{b_{i_1}b_{i_2}\dots b_{i_k}, b_{i_2}\dots b_{i_k}b_{i_1},\dots\}$ is, by definition, $\mathrm{sign}(b_{i_1})+\cdots+\mathrm{sign}(b_{i_k})=\varepsilon_{i_1}+\cdots+\varepsilon_{i_k}=j$.

So, we have a bijection between words on A, which have standard signed permutation of cycle type λ, and multi-sets of primitive necklaces with $n_i(j)$ necklaces of sign j and length i.

Then PN_λ, the generating function for multi-sets of primitive necklaces with $n_i(j)$ necklaces of length i and sign j, is

$$\mathrm{PN}_\lambda=\sum_{\mathrm{ct}(\mathrm{sts}(w))=\lambda}\mathrm{ev}(w) \tag{9}$$

(recall that $\mathrm{ct}(\alpha)$ is the cycle type of a signed permutation α).

There also exists a natural and evaluation preserving bijection, *pw*, between primitive necklaces and Lyndon words. We call the sign of a word, the sum of the signs of its letters, with multiplicities. If $h=\mathrm{pw}(\rho)$, where h is a Lyndon word and ρ is primitive necklace, then the signs and the lengths of h and ρ are equal because pw preserves the evaluation.

If we take a total order $\leqslant$ on the set of Lyndon words on A, we then have

$$\mathrm{PN}_\lambda=\sum\mathrm{ev}(h_1)\dots\mathrm{ev}(h_k)=\sum\mathrm{ev}(h_1\dots h_k), \tag{10}$$

where the h_l are Lyndon words such that there is $n_i(j)$ h_l of length i and sign j and $h_1\leqslant h_2\cdots\leqslant h_k$.

It is well known that Lyndon words index a basis $\{P_h \mid h \text{ is a Lyndon word}\}$ of the free Lie algebra on A, and that the polynomial P_h is finely homogeneous of the same partial degree as h in each letter.

The generating function of U_λ is then

$$T_\lambda=\sum_{h_1\leqslant\cdots\leqslant h_k}\mathrm{ev}(h_1\dots h_k) \tag{11}$$

where the number of h_l of length i and sign j equals $n_i(j)$.

By Eqs. (10) and (11), we have the first claim of Theorem 16 ($\mathrm{PN}_\lambda=T_\lambda$).

We also have, by relations (9)–(11),

$$T_\lambda=\sum_{\mathrm{ct}(\mathrm{sts}(w))=\lambda}\mathrm{ev}(w).$$

Thus

$$T_\lambda = \sum_{\mathrm{ct}(\sigma)=\lambda} \sum_{\mathrm{sts}(w)=\sigma} \mathrm{ev}(w). \tag{12}$$

We now need the following result

Lemma 18. *If* $\mathrm{ct}(\sigma)=\lambda$ *then* $\mathrm{ct}(\tilde{\sigma})=\mathrm{ct}(\sigma^{-1})=\lambda^{-}$ *where* $\lambda^{-}(j)=\lambda(-j)$ *for* $j\in\mathbb{Z}/p\mathbb{Z}$.

Proof. If

$$\begin{pmatrix} i_1 & \cdots & i_{k-1} & i_k \\ (i_2,\varepsilon_{i_1}) & \cdots & (i_k,\varepsilon_{i_{k-1}}) & (i_1,\varepsilon_{i_k}) \end{pmatrix}$$

is a cycle of σ then

$$\begin{pmatrix} i_1 & \cdots & i_{k-1} & i_k \\ (i_2,-\varepsilon_{i_1}) & \cdots & (i_k,-\varepsilon_{i_{k-1}}) & (i_1,-\varepsilon_{i_k}) \end{pmatrix}$$

and

$$\begin{pmatrix} i_k & i_{k-1} & \cdots & i_1 \\ (i_{k-1},-\varepsilon_{i_{k-1}}) & (i_{k-2},-\varepsilon_{i_{k-2}}) & \cdots & (i_k,-\varepsilon_{i_k}) \end{pmatrix}$$

are the corresponding cycles of $\tilde{\sigma}$ and σ^{-1}, respectively; so to each cycle of length k and sign j of σ correspond cycles of length k and sign $-j$ of $\tilde{\sigma}$ and σ^{-1}. □

From Eq. (12) and Lemma 18, one has

$$T_\lambda = \sum_{\mathrm{ct}(\sigma)=\lambda^{-}} \sum_{\mathrm{csts}(w)=\sigma} \mathrm{ev}(w).$$

By Lemma 11 and Lemma 18 again,

$$T_\lambda = \sum_{\mathrm{ct}(\sigma)=\lambda^{-}} F_{\mathrm{CS}(\sigma^{-1})} = \sum_{\mathrm{ct}(\sigma)=\lambda} F_{\mathrm{CS}(\sigma)}$$

and we have the second assertion of Theorem 16.

Theorem 6 implies immediately the third part of Theorem 16. □

5. Application to the hyperoctahedral groups

The hyperoctahedral groups are the groups obtained when $p=2$. As usual, these groups shall be denoted by B_n. The group B_n, as a Coxeter group embedded in $S_{\{-n,\dots,-1,0,1,\dots,n\}}$, is generated by $\{r_0,r_1,\dots,r_{n-1}\}$ where $r_0=(1,-1)$ and $r_i=(i,i+1)$ $(-i,-i-1)$ for every $1\leqslant i\leqslant n-1$.

From this presentation, one has a natural notion of descent set; this notion has been defined by Solomon (see [23]), so we shall say that *the Solomon descent set* of $\sigma\in B_n$ is the set

$$\mathrm{des}(\sigma)=\{i \mid 0\leqslant i\leqslant n-1,\ \sigma(i)>\sigma(i+1)\}.$$

According to the terminology of Foata and Schützenberger [7], the Solomon descent set of σ^{-1} will be called the *idown set* of σ.

Let I_n be the set $\{0,\dots,n-1\}$; if $K\subseteq I_n$ we denote by W_K the subgroup of B_n generated by the r_k, $k\in K$.

In the group algebra $\mathbb{Q}[B_n]$ of B_n over $\mathbb{Q}$, we define, for all $K\subseteq I_n$, the two idempotents

$$\xi_K=\frac{1}{|W_K|}\sum_{w\in W_K} w, \tag{13}$$

$$\eta_K=\frac{1}{|W_K|}\sum_{w\in W_K} \varepsilon(w)w, \tag{14}$$

where ε is the character of B_n such that $\varepsilon(r_i)=-1$ for $i\in I_n$. If $K\subseteq I_n$, let ψ_K (resp. ϕ_K) be the character afforded by the left ideal $\mathbb{Q}[B_n]\xi_K\eta_{I_n\setminus K}$ (resp. $\mathbb{Q}[B_n]\xi_K$). The following proposition is due to Solomon [23].

Proposition 19. *If K is a subset of I_n we have:*
(i) *For all $g\in B_n$, $\psi_K(g)=\varepsilon(g)\psi_{I_n\setminus K}(g)$.*
(ii) *The number of signed permutations having descent set K is equal to the dimension of $\psi_{I_n\setminus K}$.*
(iii) *The character ϕ_K is the character of G induced by the trivial character of W_K.*
(iv) $\psi_K=\sum_{K\subseteq J\subseteq I_n}(-1)^{|J\setminus K|}\phi_K$.

We prove the next result, which extends to the case of the hyperoctahedral group the result of Gessel [9].

Theorem 20. *Let K be a subset of I_n and $\Pi\subseteq B_n$. Let g be the quasi-symmetric series of Π. If g is symmetric then the number of elements of Π having descent set K is*

$$\langle\omega_1(\mathrm{ch}(\psi_{I_n\setminus K})),g\rangle.$$

The proof of this result is based on several lemmas and definitions.

For any group G, any subgroup H of G and any character ψ of H, we denote by $\psi\uparrow_H^G$ the character of G induced by ψ. For any group G, 1_G denotes the trivial character of G. We will also write $1\uparrow_H^G$ for $1_H\uparrow_H^G$. This should cause no confusion.

For any $g\in B_n$, we denote the cycle type $\mathrm{ct}(g)$ of g by a pair of partitions $(\lambda(g),\mu(g))$ where $\lambda(g)=\mathrm{ct}(g)(0)$ and $\mu(g)=\mathrm{ct}(g)(1)$. We shall also write X and $\bar{X}$ instead of X^0 and X^1, respectively; A is then the union of X and $\bar{X}$.

Lemma 21. *For any integer n one has:*
(i) $\mathrm{ch}(1_{B_n})=h_n(X)$,
(ii) $\mathrm{ch}(1\uparrow_{S_n}^{B_n})=h_n(A)$.

Proof. (i) The only primitive root of 1 of order 2 is -1, so we have

$$\mathrm{ch}(1_{B_n}) = \frac{1}{|B_n|} \sum_{\substack{\sigma\in B_n\\ (\lambda,\mu)=\mathrm{ct}(\sigma)}} (p_{\lambda_1}(X) + p_{\lambda_1}(\bar{X}))\dots(p_{\mu_1}(X) - p_{\mu_1}(\bar{X}))\dots$$

Observe that the number of elements of B_n of cycle type (λ,μ) is

$$\frac{n!2^n 2^{-l(\lambda)}2^{-l(\mu)}}{z_\lambda z_\mu},$$

where for any partition $\tau = 1^{n_1}2^{n_2}\dots$ one has $z_\tau = \prod_{i\geqslant 1} i^{n_i} n_i!$ (see [13, p. 47]).
Thus

$$\mathrm{ch}(1_{B_n}) = \sum_{|\lambda|+|\mu|=n} \frac{(p_{\lambda_1}(X) + p_{\lambda_1}(\bar{X}))\dots(p_{\mu_1}(X) - p_{\mu_1}(\bar{X}))\dots}{z_\lambda z_\mu 2^{l(\lambda)}2^{l(\mu)}}$$

$$= \sum_{|\tau|=n} \sum_{\lambda\cup\mu=\tau} \frac{(p_{\lambda_1}(X) + p_{\lambda_1}(\bar{X}))\dots(p_{\mu_1}(X) - p_{\mu_1}(\bar{X}))\dots}{z_\lambda z_\mu 2^{l(\lambda)}2^{l(\mu)}}.$$

By considering corresponding parts in λ and μ one obtains

$$\mathrm{ch}(1_{B_n}) = \sum_{|\tau|=n} \prod_{i\geqslant 1} \sum_{k=0}^{n_i} \frac{1}{2^{n_i} i^{n_i} k!(n_i-k)!} (p_i(X) + p_i(\bar{X}))^k (p_i(X) - p_i(\bar{X}))^{n_i-k}$$

$$= \sum_{|\tau|=n} \prod_{i\geqslant 1} \frac{1}{2^{n_i} i^{n_i} n_i!} (2p_i(X))^{n_i}$$

$$= \sum_{\tau\vdash n} z_\tau^{-1} p_\tau(X) = h_n(X) \quad \text{(see [16, p. 17])}.$$

(ii) This proof is essentially the same as that of (i); it is based on the fact that for any $g\in B_n$ one has

$$\begin{cases} 1\uparrow_{S_n}^{B_n}(g) = 0 & \text{if } \mu(g) \neq \emptyset, \\ 1\uparrow_{S_n}^{B_n}(g) = 2^{l(\mathrm{ct}(g))} & \text{if } \mu(g) = \emptyset. \end{cases} \qquad \square$$

We shall denote by $\hat{K}$ the set $(I_n\backslash K)\backslash\{0\}$. We define the *composition* $C(K)$ *of* K by $C(K) = (c_1,\dots,c_i)$ if $\hat{K} = \{c_1, c_1+c_2, \dots, c_1+\dots+c_{i-1}\}$ and $c_1+\dots+c_i = n$.
The following lemma expresses the ϕ_K and the ψ_K in terms of the $\chi_{[T,m]}$.

Lemma 22. *Let K and J be subsets of I_n, Hence*

$$\phi_K = \begin{cases} \sum_{K \subseteq L \subseteq I_n \setminus \{0\}} \sum_{m:[n] \to \{0,1\}} \chi_{[\hat{L},m]} & \text{if } 0 \notin K, \\ \sum_{K \subseteq L \subseteq I_n} \sum_{\substack{m:[n] \to \{0,1\} \\ m(1)=0}} \chi_{[\hat{L},m]} & \text{if } 0 \in K, \end{cases}$$

$$\psi_J = \begin{cases} \sum_{\substack{m:[n] \to \{0,1\} \\ m(1)=0}} \chi_{[\hat{J},m]} & \text{if } 0 \in J, \\ \sum_{\substack{m:[n] \to \{0,1\} \\ m(1)=1}} \chi_{[\hat{J},m]} & \text{if } 0 \notin J. \end{cases}$$

Proof. Let $K \subseteq I_n$.

If $0 \notin K$, $W_K = S_{c_1} \times S_{c_2} \times \cdots \times S_{c_i}$ where $C(K) = (c_1, \ldots, c_i)$, then $\phi_K = 1 \uparrow_{W_K}^{B_n} = 1 \uparrow_{S_{c_1} \times \cdots \times S_{c_i}}^{B_n} = 1 \uparrow_{S_{c_1} \times \cdots \times S_{c_i}}^{B_{c_1} \times \cdots \times B_{c_i}} \uparrow_{B_{c_1} \times \cdots \times B_{c_i}}^{B_n}$.

Thus $\phi_K = 1 \uparrow_{S_{c_1}}^{B_{c_1}} \ldots . 1 \uparrow_{S_{c_i}}^{B_{c_i}}$, so by Lemma 21 (ii) we have

$$\mathrm{ch}(\phi_K) = h_{c_1}(A) \ldots h_{c_i}(A).$$

This shows that $\mathrm{ch}(\phi_K)$ is the sum of evaluations of all the 2-semi-standard tableaux of shapes

$$\begin{array}{llll} \overset{c_1}{\bullet \ldots | \ldots \bullet} & & & \\ & \overset{c_2}{\bullet \ldots | \ldots \bullet} & & \\ & & \ddots & \\ & & & \overset{c_i}{\bullet \ldots | \ldots \bullet} \end{array}$$

where in each row the bar can move from left to right.

We then have

$$\begin{aligned} \mathrm{ch}(\phi_K) &= \sum_v h_{(C(K),v)} \\ &= \sum_{J \subseteq \hat{K}} \sum_{m:[n] \to \{0,1\}} S_{[J,m]} \quad \text{by Lemma 13} \\ &= \sum_{K \subseteq L \subseteq I \setminus \{0\}} \sum_{m:[n] \to \{0,1\}} S_{[\hat{L},m]}. \end{aligned}$$

If $0 \in K$, one has $W_K = B_{c_1} \times S_{c_2} \times \cdots \times S_{c_i}$, and a similar argument shows that

$$\mathrm{ch}(\phi_K) = h_{c_1}(X) h_{c_2}(A) \ldots h_{c_i}(A).$$

Then $\mathrm{ch}(\phi_K)$ is the sum of evaluations of all the 2-semi-standard tableaux of shapes

$$\begin{array}{llll}
\overset{c_1}{|\bullet\ldots\bullet} & & & \\
 & \overset{c_2}{\bullet\ldots|\ldots\bullet} & & \\
 & & \ddots & \\
 & & & \overset{c_i}{\bullet\ldots|\ldots\bullet}
\end{array}$$

where in each row but the first, the bar can move from left to right.

So we have

$$\mathrm{ch}(\phi_K)=\sum_{K\subseteq L\subseteq I}\ \sum_{\substack{m:[n]\to\{0,1\}\\ m(1)=0}} S_{[\hat{L},m]}.$$

The equations for ψ_K are deduced from the formula $\phi_K=\sum_{K\subseteq J\subseteq I}\psi_J$, due to Solomon [23], and by straightforward calculation. □

We define, for any alphabet Y, the isometric automorphism ω of $\Lambda(Y)$ by

$$\omega(s_\lambda(Y))=s_{\lambda'}(Y),$$

where λ' means the conjugate partition of λ, (see [16]). Let ω_0 and ω_1 be the isometric automorphisms of $\Lambda(X)\otimes\Lambda(\bar{X})$ defined by

$$\omega_0(s_\lambda(X)\otimes s_\mu(\bar{X}))=\omega(s_\lambda(X))\otimes s_\mu(\bar{X}), \tag{15}$$

$$\omega_1(s_\lambda(X)\otimes s_\mu(\bar{X}))=s_\lambda(X)\otimes \omega(s_\mu(\bar{X})). \tag{16}$$

We shall denote by ω_0' and ω_1' the correspondant automorphisms of the algebra of characters on B_n.

Let σ be an element of B_n having Solomon descent set K and signed descent set $[S,m]$. The following link between K and $[S,m]$ holds:

- $i\in K \Leftrightarrow (i\in S$ and $m(i)=0$ or $m(i+1)=0)$ or

$$(i\notin S \text{ and } m(i)=1 \text{ and } m(i+1)=1) \tag{17}$$

- $0\in K\Leftrightarrow m(1)=1$.

In fact, one has $-n<-(n-1)<\cdots<-1<1<\cdots<n$, and the order used to define the signed descent set is $(1,1)<_r(2,1)<_r\cdots<_r(n,1)<_r(1,0)<_r\cdots<_r(n,0)$. We then have to reverse the order on negative elements. This exchange rises and descents in the negative segments.

Moreover, the equivalence in (17) shows that if σ has signed descent set $[S,m]$ then σ has Solomon descent set K. We shall denote by $\mathrm{Sig}(K)$ the set of signed sets $[S,m]$ which verify (17) for K. The following lemma comes from the fact that applying ω_1 to $S_{[T,m]}$ corresponds to reversing the order on the negative elements, i.e., to exchanging rises and descents in negative segments. So that one has $S_{[K,m]}=\omega_1(S_{[S,m]})$

for $[S,m] \in \mathrm{Sig}(K)$ (if necessary, we remove 0 from K). Lemma 22 allows us to conclude.

Lemma 23. *For all $K \subseteq I_n$ one has*

$$\omega_1(\mathrm{ch}(\psi_{I_n \setminus K})) = \sum_{[S,m] \in \mathrm{Sig}(K)} S_{[S,m]}.$$

Proof of Theorem 20. Theorem 15 and the equivalence in (17) show that the number of elements of Π having Solomon descent set K is equal to

$$\left\langle g, \sum_{[T,m] \in \mathrm{Sig}(K)} S_{[T,m]} \right\rangle,$$

where g is the quasi-symmetric generating series of Π.

But Lemma 23 claims that

$$\omega_1(\mathrm{ch}(\psi_{I_n \setminus K})) = \sum_{[S,m] \in \mathrm{Sig}(K)} S_{[S,m]}.$$

The number of elements of Π having Solomon descent set K is then

$$\langle g, \omega_1(\mathrm{ch}(\psi_{I_n \setminus K})) \rangle. \qquad \square$$

We can now prove the two following corollaries which extend Theorem 2.1 of [10] and Theorem 5 of [9] to the case of hyperoctahedral groups.

Corollary 24. *The number of signed permutations having cycle type (λ, μ) and Solomon descent set $K \subseteq I_n$ is*

$$\langle \chi_{(\lambda,\mu)}, \omega_1'(\psi_{I_n \setminus K}) \rangle.$$

Proof. This is an immediate consequence of the fact that ω_1' is an isometric involution of R and of Theorems 17 and 20. $\square$

Corollary 25. *Let K and J be subsets of I_n. Then the number of elements of B_n having descent set K and idown set J is*

$$\langle \psi_{I_n \setminus K}, \psi_{I_n \setminus J} \rangle = \langle \psi_K, \psi_J \rangle.$$

Proof. The proof of Theorem 7 shows that the quasi-symmetric generating series of the elements σ of B_n such that $(\tilde{\sigma})^{-1}$ has signed descent set $[T,f]$ is $S_{[T,f]}$. But in our case, one has $\tilde{\sigma} = \sigma$. So $S_{[T,f]}$ is the quasi-symmetric generating series of the elements σ of B_n such that σ^{-1} has signed descent set $[T,f]$. The quasi-symmetric generating series of the elements of B_n whose inverse has Solomon descent set K is then

$$\sum_{[S,m] \in \mathrm{Sig}(K)} S_{[S,m]}.$$

Theorem 20 and Lemma 23 achieve the proof of the corollary.

For the right-hand side, we use the equality $\varepsilon(g)\psi_K(g)=\psi_{I_n\setminus K}(g)$ (Proposition 19 (i)), and the fact that $\varepsilon(g)\varepsilon(g^{-1})=1$. $\square$

6. Alternating signed permutations and trigonometric symmetric functions

A *rising alternating* (resp. *falling alternating*) signed permutation σ is a signed permutation having descent set $K_n^r=\{1,3,\ldots\}\subset I_n$ (resp. $K_n^f=\{0,2,\ldots\}\subset I_n$).

Example. The signed permutation $\sigma=3\,1\,4\,\bar{6}\,7\,\bar{2}\,5$ is a rising alternating element of B_7 and $\sigma=\bar{1}\,4\,3\,7\,\bar{6}\,\bar{2}\,\bar{5}$ is a falling alternating element of B_7.

Let b_n denotes the number of rising alternating signed permutations in B_n. If one takes $b_0=1$, one then has, see [25, p. 35],

$$\sum_{n\geqslant 0}\frac{b_n}{n!}t^n=\frac{\sin x+\cos x}{\cos 2x}. \tag{18}$$

For any set of variables Y, we define, according to Désarménien [4], the symmetric cosine and sine by

$$\mathrm{COS}_Y(t)=\sum_{n\geqslant 0}(-1)^n h_{2n}(Y)t^{2n}, \tag{19}$$

$$\mathrm{SIN}_Y(t)=\sum_{n\geqslant 0}(-1)^n h_{2n+1}(Y)t^{2n+1}, \tag{20}$$

where $h_n(Y)$ is the complete symmetric function on X.

We then have the following theorem, which extends [4, Proposition 4.2] and gives symmetric analogs to Eq. (18).

Theorem 26. *If $H_X(t)=\sum_{r\geqslant 0}h_r(X)t^r$, then one has the four relations*

$$B_o^r(t)=\sum_{n\geqslant 0}\mathrm{ch}(\psi_{K_{2n+1}^r})t^{2n+1}=H_X(\mathrm{i}t)H_X(-\mathrm{i}t)\frac{\mathrm{SIN}_{\bar{X}}(t)}{\mathrm{COS}_A(t)},$$

$$B_o^f(t)=\sum_{n\geqslant 0}\mathrm{ch}(\psi_{K_{2n+1}^f})t^{2n+1}=\frac{\mathrm{SIN}_X(t)}{\mathrm{COS}_A(t)},$$

$$B_p^r(t)=1+\sum_{n\geqslant 1}\mathrm{ch}(\psi_{K_{2n}^r})t^{2n}=\frac{\mathrm{COS}_X(t)}{\mathrm{COS}_A(t)},$$

$$B_p^f(t)=1+\sum_{n\geqslant 1}\mathrm{ch}(\psi_{K_{2n}^f})t^{2n}=H_X(\mathrm{i}t)H_X(-\mathrm{i}t)\frac{\mathrm{COS}_{\bar{X}}(t)}{\mathrm{COS}_A(t)}.$$

The following lemma gives some results on trigonometric symmetric functions; these formulae give some symmetric analogs to classical formulae on trigonometric functions.

Acknowledgements

This article is a part of author's dissertation [19], written under the supervision of Christophe Reutenauer. The author thanks the LACIM and the ISM for logistic and financial support during his studies at Montréal.

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Lemma 27. *One has the following three relations*:
(i) $\mathrm{COS}_X(t)^2+\mathrm{SIN}_X(t)^2=H_X(\mathrm{i}t)H_X(-\mathrm{i}t)$,
(ii) $\mathrm{COS}_A(t)=\mathrm{COS}_X(t)\mathrm{COS}_{\bar{X}}(t)-\mathrm{SIN}_X(t)\mathrm{SIN}_{\bar{X}}(t)$,
(iii) $\mathrm{SIN}_A(t)=\mathrm{COS}_X(t)\mathrm{SIN}_{\bar{X}}(t)-\mathrm{SIN}_X(t)\mathrm{COS}_{\bar{X}}(t)$.

Proof. The following facts are easy to verify

$$\mathrm{COS}_X(t)=\frac{H_X(\mathrm{i}t)+H_X(-\mathrm{i}t)}{2}, \tag{21}$$

$$\mathrm{SIN}_X(t)=\frac{H_X(\mathrm{i}t)-H_X(-\mathrm{i}t)}{2\mathrm{i}}, \tag{22}$$

$$h_r(A)=\sum_{p+q=r} h_p(X)h_q(\bar{X}). \tag{23}$$

The proof of the lemma is then a straightforward calculation. □

Proof of Theorem 26. We have, from Proposition 19(iv),

$$\begin{aligned}\mathrm{ch}(\psi_{K^r_{2n+1}}) &= \sum_{K^r_{2n+1}\subseteq J\subseteq I_{2n+1}} (-1)^{|J\setminus K^r_{2n+1}|}\,\mathrm{ch}(\phi_J)\\ &=\sum_{k=0}^{n}\sum_{\substack{K^r_{2n+1}\subseteq J\subset I_{2n+1}\\ 2k=max(I_{2n+1}\setminus J)}} (-1)^{|J\setminus K^r_{2n+1}|}\,\mathrm{ch}(\phi_J)+(-1)^{2n+1-|K^r_{2n+1}|}\,\mathrm{ch}(\phi_{I_{2n+1}}).\end{aligned}$$

But $|K^r_{2n+1}|$ is equal to n. Furthermore, if $2k$ is the greatest element of I_{2n+1} not in J, one has

$$C(J)=(c_1,\dots,c_{i-1},2n-2k+1)$$

and $C(J)$ has at least two parts. Hence J is the disjoint union of $J'\in I_{2k}$ and $2k+1,\dots,$ $2n$, and the composition of J' is $(c_1,\dots,c_{i-1})$. Then, we have

$$\begin{aligned}\mathrm{ch}(\psi_{K^r_{2n+1}}) &= \sum_{k=0}^{n}\sum_{K^r_{2k}\subseteq J'\subset I_{2k}} (-1)^{|J'\setminus K^r_{2k}|}\,\mathrm{ch}(\phi_{J'})(-1)^{n-k}h_{2n-2k+1}(A)\\ &\quad+(-1)^{n+1}h_{2n+1}(X)\\ &=\sum_{k=0}^{n}(-1)^{n-k}h_{2n-2k+1}(A)\,\mathrm{ch}(\psi_{K^r_{2k}})+(-1)^{n+1}h_{2n+1}(X).\end{aligned}$$

This shows that $B^r_o(t)=\mathrm{SIN}_A(t)B^r_e(t)-\mathrm{SIN}_X(t)$. By the same method we find

$$B^r_e(t)\,\mathrm{COS}_A(t)=\mathrm{COS}_X(t),$$

$$B^f_e(t)=\mathrm{SIN}_A(t)B^f_o(t)+\mathrm{COS}_X(t),$$

$$B^f_o(t)\,\mathrm{COS}_A(t)=\mathrm{SIN}_X(t).$$

The resolution of these equations gives Theorem 26. □

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